Integration by Method of Substitution

Study Material - Some integrands which are not in standard form can be reduced to one or other of the standard form by suitably changing the ~~st~~ independent variable.

Method of Integration

Form I - $\int f\{g(x)\}\, g'(x)\, dx$

for put $g(x) = z$

$g'(x)\, dx = dz$

$\Rightarrow \int f(z)\, dz.$

Form II — $\int \frac{f'(x)}{f(x)}\, dx$ put $f(x) = z$

$\Rightarrow f'(x)\, dx = dz$

Form III - $\int \frac{f'(x)}{\sqrt{f(x)}}\, dx$ put $f(x) = z$

$f'(x)\, dx = dz$

Form IV :-> $\int \sin[f(x)]\, f'(x).dx$

put $f(x) = z$

$\Rightarrow f'(x)\, dx = dz$

Form V — $\int [f(x)]^n\, f'(x)\, dx$

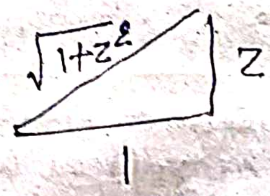
put $f(x) = z \Rightarrow [f(x)]^n = z^n$

$f'(x)\, dx = dz$

Form :-> $\int \sin^m x \cos^n x\, dx$

Case - I If $m + n = -ve$ Even No.

put $\tan x = z$



$\Rightarrow \sin x = \dfrac{z}{(1+z^2)^{1/2}}$

$\cos x = \dfrac{1}{(1+z^2)^{1/2}}$

$\sec^2 x = 1+z^2$

Case 2

If. $m+n = +ve$ even No

Also $m = +$ even No

$n = +$ even No

Then use Multiple angle Formula

$\sin^2 kx = \frac{1}{2}[1-\cos 2kx]$

$\cos^2 kx = \frac{1}{2}[1+\cos 2kx]$

Case 3. If m is $+$odd

then we write $I = \int \sin^{m-1} x \cos^n x \sin x \, dx$

Convert $\sin^{n-1} x$ in ~~convert~~ $\sin^{n-1} x$ Terms of $\cos x$ by using $\sin^2 x = 1-\cos^2 x$

put $\cos x = z$

$\sin x \, dx = -dz$

If n is odd

then we write.

$\int \sin^m x \cos^{n-1} x \cos x \, dx$

Convert $\cos^{n-1} x$ in Terms of $\sin x$

by using $\cos^2 x = 1-\sin^2 x$

put $\sin x = z$

$\cos x \, dx = dz$

Form - $\int \sec^m x \tan^n x \, dx$

$\int \sec^{m-1} x \tan^{n-1} x \sec x \tan x \, dx$	$\int \sec^{m-2} x \tan^n x \sec^2 x \, dx$
put $\sec x = z$ $\Rightarrow \sec x \tan x \, dx = dz$	put $\tan x = z$ $\sec^2 x \, dx = dz$
Convert $\tan^{n-1} x$ in Term of $\sec x$ it is used only $n-1 =$ even	Convert $\sec^{m-2} x$ in Terms of $\tan x$ an is possible Only $m-2 =$ even

Same for $\int \cot^m x \csc^n x \, dx$

Form :→ Function of the form $f(ax+b)$

put $ax+b = z$

put $ax+b = z^2$ if $ax+b$ occurs under the sign of square root.

$ax+b = z^3$ if $ax+b$ occurs under the sign of cube root.

Trigonometric substitution

$\sqrt{a^2+x^2}$

put $x = a\tan\theta \Rightarrow \tan\theta = x/a$

$dx = a\sec^2\theta \, d\theta \quad \Rightarrow \theta = \tan^{-1} x/a$

$a^2+x^2 = a^2\tan^2\theta.$

$\sqrt{a^2-x^2}$

put $x = a\sin\theta \quad \sin\theta = x/a$

$\Rightarrow a^2-x^2 = a^2\cos^2\theta \quad \theta = \sin^{-1} x/a$

$dx = a\cos\theta \, d\theta$

$\sqrt{x^2-a^2}$

put $x = a\sec\theta \Rightarrow \theta = \sec^{-1} x/a$

$dx = a\sec\theta\tan\theta \, d\theta$

$x^2-a^2 = a^2\tan^2\theta$

$\sqrt{a-x}$ or $\sqrt{a+x}$ or $\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$

put $x = a\cos 2\theta$

$dx = -2a\sin 2\theta \, d\theta$

$a-x = a(1-\cos 2\theta) = 2a\sin^2\theta$

$a+x = a(1+\cos 2\theta) = 2a\cos^2\theta$

$\theta = \frac{1}{2}\cos^{-1} x/a$

Evaluate 1. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

$\int \sin(g(x))\, g'(x)\,dx$

$\int \cos(g(x))\, g'(x)\,dx$

put $g(x) = z$

$g'(x)\,dx = dz$

put $\sqrt{x} = z$

$\frac{1}{2\sqrt{x}}\,dx = dz$

$\frac{1}{\sqrt{x}}\,dx = 2\,dz$

$I = 2\int \cos z\,dz = 2\sin z + C$

$= 2\sin\sqrt{x} + C$

2. $I = \int \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx$

Form $\int \sin(g(x))\,dx$

put $g(x) = z$

$g'(x)\,dx = dz$

put $\frac{1}{x} = z$

$-\frac{1}{x^2}\,dx = dz \Rightarrow \frac{1}{x^2}\,dx = -dz$

$= -\int \sin z\,dz = \cos z + C$

$= \cos\left(\frac{1}{x}\right) + C$

3. $\int (1+\cos x)^5 \sin x\,dx$

form $\int [f(x)]^n f'(x)\,dx$

put $f(x) = z$

$f'(x)\,dx = dz$

put $1+\cos x = z$

$\sin x\,dx = -dz$

$= -\int z^5\,dz = -\frac{z^6}{6} + C$

$= -\frac{(1+\cos x)^6}{6} + C$

4. $\int \frac{\sec^2 x}{\sqrt{1+\tan x}}\,dx$

$= \int [1+\tan x]^{-1/2} \sec^2 x\,dx$

Put $1+\tan x = z$ $\quad$ form $\int [f(x)]^n f'(x)\,dx$

$\sec^2 x\,dx = dz$

$\therefore I = \int z^{-1/2}\,dz = 2z^{1/2} + c$

$= 2\sqrt{1+\tan x} + c$

5. $I = \int \frac{\sec^2(\log x)}{x}\,dx$ $\quad$ $\int f(g(x))\,g'(x)\,dx$

put $g(x) = z$

$\Rightarrow g'(x)\,dx = dz$

put $\log x = z$

$\frac{1}{x}\,dx = dz$

$= \int \sec^2 z\,dz = \tan z + c$

$= \tan(\log x) + c$

6. $\int \frac{1+\cos x}{\sqrt[3]{x+\sin x}}\,dx$ $\quad$ form $[f(x)]^n f'(x)\,dx$

$= \int [x+\sin x]^{-1/3}(1+\cos x)\,dx$ $\quad$ put $f(x) = z$

$\Rightarrow f'(x)\,dx = dz$

put $x+\sin x = z$

$(1+\cos x)\,dx = dz$

$= \int z^{-1/3}\,dz = \frac{z^{-1/3+1}}{-1/3+1} + c$

$= \frac{3}{2} z^{2/3} + c$

$= \frac{3}{2}(x+\sin x)^{2/3} + c$

Evaluate

I $\displaystyle\int \frac{1-\sin 2x}{x+\cos^2 x}\,dx$ $\qquad$ Form $\displaystyle\int \frac{f'(x)}{f(x)}\,dx$

put $f(x)=z$

put $x+\cos^2 x = z$ $\qquad$ $f'(x)\,dx = dz$

$[1+2\cos x(-\sin x)]\,dx = dz$

$\Rightarrow [1-\sin 2x]\,dx = dz$ $\qquad$ As $2\sin x\cos x = \sin 2x$

$= \displaystyle\int \frac{dz}{z} = \log z + c.$

$= \log(x+\cos^2 x) + c$

II $\displaystyle\int \frac{\sin x+\cos x}{\sin x-\cos x}\,dx$ $\qquad$ form $\displaystyle\int \frac{f'(x)}{f(x)}\,dx$

put $\sin x - \cos x = z$

$[\cos x + \sin x]\,dx = dz$

$= \displaystyle\int \frac{dz}{z} = \log.|z|$

$= \log|\sin x - \cos x| + c$